Teaching the characteristics of yield response with the Mitscherlich equation using computers[1]

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ABSTRACT

Students studying plant and soil sciences must understand the relationships of how plant variables such as yields and composition change with fertilizer inputs. The Mitscherlich equation is a traditional and useful expression of such a relationship. In this paper, some of the background and development of modified but general Mitscherlich equations with one or more independent variables are presented. Interactive computer programs in the BASIC and FORTRAN languages are described and listed. Use of the programs is illustrated to allow students to work with these equations with a view toward developing competence in fitting of yield curves to observed data.

Additional index words: **Yield equation, Computer-assisted instruction.**

ONE of the oldest, most widely known, and most studied concepts in the field of soil fertility is the Mitscherlich equation. Although empirically derived, it has shown wide applicability to data obtained from fertilization of crops. Although much work was done with the Mitscherlich equation in the first half of this century, only a few reports have been evident since 1960 (13). Part of the neglect of this equation is due to its relative mathematical complexity and to the difficulty of statistical evaluation of general forms of the equation. The first step in increasing its use, however, is the education of students in the characteristics and applications of this equation.

The purpose of this paper is to a) provide some background on the Mitscherlich equation, b) present the derivation of a modified form of the equation, c) describe a computer program to assist in familiarizing students with the equation, d) generalize the Mitscherlich equation to more than one independent variable, and e) present a computer program for the use of students in studying multifactor equations.

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HISTORICAL BACKGROUND

The Mitscherlich equation is an expression of the principle that has been described as the Law of Diminishing Increments as applied to the effect of fertilization on crop yields. It was first proposed by Mitscherlich in 1909 (18). In the same article, Mitscherlich credits Wollny with the generalization of the equation to apply to growth factors other than fertilizers. Large amounts of data have been collected demonstrating the applicability of this equation to yield data (11, 12, 15, 16, 17, 19, 23, 24). The most common form of this equation is $y = A[1 - \exp(-cx)]$ where y is the yield, x is the fertilizer rate, A is the yield possibility, and c is a proportionality constant.

The equation was criticized early because Mitscherlich had claimed that the parameter c in the equation was constant for all crops and all cropping conditions. Also, some of the interpretations of the Mitscherlich equation made by Willcox (23, 24) were called into question by van der Paauw (20, 21) and others. One of the most intense interchanges in soil science occurred on this point between Willcox (25, 26, 27, 28, 29) and Black and co-workers (7, 8, 9, 22). These papers give much insight into the controversy.

Baule (5) attempted to resolve the observed discrepancies in the constant c by proposing the conversion of the weight of applied nutrients to Baule units, the amounts of fertilizer needed to give 50% of the maximum yield. This approach was carried further by Cooper and Hall (12) to develop what they called the balanced-baulic-poundage ratio and showed the consistency of these ratios in legume crops. Apparently this line of research was not carried further.

Another approach was pursued by Balba, Bray, and coworkers (1, 2, 3, 4, 10, 11). This equation is usually expressed as $y = A[1 - \exp(-cx - c_1b)]$ where A and c are defined as before, c_1 is the proportionality constant for the soil nutrient in question and b is the amount of available soil nutrient. They provided for the nutrient supplied from the soil with an additional term in the exponent of the equation. This modified equation has been productive in several lines of research although it is somewhat more complex in the sense that two additional parameters are added. There is also some ambiguity of the appropriate value of the constant b, the amount of available nutrient in the soil. A modification of this equation was used by Englestad and Khasawneh (13) in fertilizer evaluation.

The problem of determination of the constants has been dealt with in a number of ways. Many involve estimation of one or more of the parameters. Other procedures suited to a specific problem have been developed. Behrens (6) has described some of the approaches to solving the problem in general and proposed a technique using the Gaussian Principle and a graphical solution. It was the most general and mathematically simple procedure although it still required the estimation of the yield with no fertilizer applied.

A MODIFIED MITSCHERLICH EQUATION

The only assumption in the basic Mitscherlich equation is that the relationship between amount of fertilizer added and the crop yield is that described by the Law of Diminishing Increments. This Law was expressed as a differential equation by Mitscherlich (18). Some of the considerations involved in this assumption are described by Fried and Broeshart (14). The differential equation is:

$$dy/dx = c(A - y) \quad [1]$$

where y is the yield, x is the amount of nutrient applied, c is the effect factor of Mitscherlich, and A is the yield possibility or the maximum yield that may be reached by

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C     MITSCHERLICH 1-DIMENSIONAL PROGRAM
C     PREPARED BY R. C. SORENSEN
C     UNIVERSITY OF NEBRASKA
      DIMENSION Y(10),X(10),DIFF(10),TY(10)
      DATA YES/'Y'/
    9 WRITE(6,10)
   10 FORMAT(' HOW MANY DATA POINTS DO YOU HAVE?')
      READ(5,*)NO
      WRITE(6,20)NO
   20 FORMAT(' ENTER THE ',I2,' VALUES OF THE DEPENDENT VARIABLE:')
      READ(5,*)(Y(N),N=1,NO)
      WRITE(6,30)NO
   30 FORMAT(' ENTER THE ',I2,' VALUES OF THE INDEPENDENT VARIABLE:')
      READ(5,*)(X(M),M=1,NO)
      A=Y(NO)
      YO=Y(1)
    8 DO11K=1,NO
      IF(Y(K).GE.A)A=Y(K)+1.
   11 CONTINUE
   32 V=1.-(YO/A)
      NL=NO-1
      ICT=0
      CSUM=0.
      DO5I=2,NL
      ICT=ICT+1
      C=(-1./X(I))*(ALOG((1./V)*(1.-(Y(I)/A))))
    5 CSUM=CSUM+C
      C=CSUM/ICT
      S=0.
      SQ=0.
      SSQY=0.
      SY=0.
      DO6J=1,NO
      SSQY=SSQY+Y(J)**2
      SY=SY+Y(J)
      TY(J)=A*(1.-V*EXP(-1.*C*X(J)))
      DIFF(J)=TY(J)-Y(J)
      S=S+DIFF(J)
    6 SQ=SQ+(DIFF(J)**2)
      CSSY=SSQY-(SY**2)/NO
      R2=1.-(SQ/CSSY)
      WRITE(6,40)YO,V,C,A,S,SQ,R2,(DIFF(L),L=1,NO)
   40 FORMAT(' SOLUTION: YO = ',F8.4,' V = ',F6.4,' C = ',F8.4,' A = ',
     1F7.2/' SUM DEV = ',F9.4,' SSQ = ',F12.4,' R2 = ',F5.3/' DEVIATIONS
     2: ',10F6.2)
      PCSF=(1.-V)*100
      WRITE(6,50)
   50 FORMAT(' ENTER NEW VALUES OF YO AND A (OR PRESS RETURN TO STOP):')
      READ(5,*,END=100)YO,A
      GOTO 32
  100 REWIND 5
      WRITE(6,60)
   60 FORMAT(' DO YOU WANT TO SEE THE PREDICTED VALUES?')
      READ(5,70)R
   70 FORMAT(A1)
      IF(R.EQ.YES)GOTO 85
   79 WRITE(6,80)
   80 FORMAT(' DO YOU WANT TO DO ANOTHER PROBLEM?')
      READ(5,70)R1
      IF(R1.EQ.YES)GOTO 9
      STOP
   85 WRITE(6,90)C,V,A,PCSF,(TY(I),I=1,NO)
   90 FORMAT(' C = ',F8.4,' V = ',F8.4,' A = ',F8.2,' PCT SFCY = ',
     1F4.1/' Y VALUES:',10F7.2)
      WRITE(6,130)(X(J),J=1,NO)
  130 FORMAT(' X VALUES:',10F7.2)
      GOTO 79
      END
```

Fig. 1. A FORTRAN program for fitting a Mitscherlich Equation to a set of data points.

the application of the nutrient being studied. This equation may be integrated to give:

$$\ln(A - y) = -cx + C \quad [2]$$

where C is an arbitrary constant of integration. Several investigators have overlooked the fact that this integration must give a natural logarithm rather than a common logarithm. Use of the common logarithm causes the incorporation of a multiplicative error of 2.303 (i.e., ln 10) in the value of c which can, nonetheless, easily be taken into account.

One boundary condition is needed to evaluate C. The point used is usually the yield with no fertilizer applied, although other approaches such as that used by Balba and Bray are also effective. If y_0 is the yield with no nutrient applied than $C = \ln(A - y_0)$ and the complete equation becomes:

$$\ln(A - y) = -cx + \ln(A - y_0) \quad [3]$$

or if expressed in the exponential form:

$$y = A[1 - (1 - y_0/A] \exp(-cx) \quad [4]$$

The quantity y_0/A, when multiplied by 100, is the percentage sufficiency as defined by Baule (5). Define the quantity v, then, such that:

$$v = 1 - y_0/A \quad [5]$$

Thus, v, when multiplied by 100, is in a sense the percentage deficiency. Substituting this expression into equation 4 yields:

$$y = A[1 - v\exp(-cx)] \quad [6]$$

It is evident that v is equal to $\exp(-c_1 b)$ in Balba and Bray's equation.

Equation 6 serves as the basis for the next discussion. Note that no assumptions have been made regarding the amount or availability of the soil nutrient.

THE COMPUTER PROGRAM

Application

The advanced soil fertility course at the University of Nebraska has, for many years, included a study of the basis and characteristics of the Mitscherlich equation. One aspect that has not been included was the fitting of this equation to observed data, owing primarily to a lack of a suitably simple and accessible way for students to perform a least-squares analysis on an equation of this type. The computer programs described herein provide that capability.

These programs will be used in Agronomy 966, an advanced level graduate course in soil fertility (15 to 20 students). For the Mitscherlich exercise, students are expected to have competence in college algebra and statistics and, if possible, some calculus. However, as a rule, less than one-third of the students have completed any calculus courses.

Students are given a problem set with which they will use the programs. First, they are asked to fit a Mitscherlich equation to a set of observed data. Second, for the same set of data, they form a family of equations varying in c and a family of equations varying in A. On the basis of the graphs of these equations and the statistical data, they determine the effects of these two constants on the goodness of fit of the equation. The third problem is fitting a set of data based on two independent variables with the computer program designed for that purpose. Fourth, the students fit the same two-variable data using polynomial regression and compare predicted values for the two equations. Owing to the nature of the facilities, the FORTRAN programs are used for all problems in this course.

As a result of this exercise, the students are expected to improve their conceptualization of the Mitscherlich equation and develop greater ability in interpreting equations developed from least-squares analysis. No student response data to this exercise have been collected. However, other exercises using computer assistance assigned in this class have been judged as very helpful by past students.

Description

Although the mathematical manipulations required by the Mitscherlich equation have value in a student's education, using a computer to make the calculations may allow the student to give more attention to the applications of the equation. With this in mind, a computer program was developed to facilitate such studies. A listing of the program in FORTRAN is provided in Fig. 1 and a listing in BASIC for microcomputers is shown in Fig. 2. A terminal session using the program in FORTRAN is shown in Fig. 3.

The session begins with a request to the student to enter the number of data points to be usd to fit the equation, the values of the dependent variable (yields), and the values of the independent variable (fertilizer amounts). The value of A is set by the program at the highest yield plus one. The yield at the lowest fertilizer rate is assigned to y_0. After the determination of v, a value of c is calculated for each point except the endpoints and these are averaged. Using these values of A, v, and c, predicted yields are determined. Information then printed includes the values of y_0, A, v, and c used in the calculations, the sum and sum of squares of deviations of the predicted from the observed data, the R^2 value, and a list of individual deviations for each yield.

On the basis of these data the student may make changes in either y_0 or A to attempt to reduce the deviations and increase the R^2 value just presented. Upon entry of the new values, a new set of constants will be calculated and a new listing of the results will be printed. The student may continue to alter y_0 and A, or both, until he or she is satisfied that the constants are sufficiently precise. Then a summary of the constants, the percent sufficiency, the fertilizer amounts and the predicted yields are printed.

A Generalized Equation and Program

In his proposal of the percentage sufficiency concept, Baule (5) generalized the Mitscherlich equation for the treatment of more than one nutrient. Yet, very limited use has been made of this possibility. The constants of the generalized equation may be defined similarly as for the one-dimensional case. The yield possibility considering all variables being studies is A. However, there will be different "effect" factors, c, and percentage deficiencies, v, for each variable. The generalized equation will be:

$$y = A\,[1 - v_1 \exp(-c_1x_1)][1 - v_2 \exp(-c_2v_2)]$$
$$[1 - v_3 \exp(-c_3x_3)] \ldots \quad [7]$$

Note that only two principles are assumed in this equation; the Law of Diminishing Increments and the percentage sufficiency concept. The boundary conditions are consistent since infinitely large values of all x-variables will give y = A and zero values for all x-variables will give a y-value equal to the product of the percentage sufficiencies. Also, if the values of all x-variables are infinitely high but one, the equation reduces to the one-dimensional Mitscherlich equation.

A computer program to study multidimensional Mitscherlich equations is necessarily more complex than one for equations of one dimension. A FORTRAN program for study of a two-dimensional equation has been developed. (A copy of the FORTRAN program is available from the author.) A BASIC program was not produced owing to the size of the program.

```
10    DIM Y(10), X(10), D2(10), T2(10)
20    Y$ = "YES"
30    PRINT"HOW MANY DATA POINTS DO YOU HAVE"
40    INPUT N2
50    PRINT"ENTER THE";N2;"VALUES OF THE DEPENDENT VARIABLE:"
60    FOR K=1 TO N2
70    INPUT Y(K)
80    NEXT K
90    PRINT"ENTER THE";N2;"VALUES OF THE INDEPENDENT VARIABLE:"
100   FOR J = 1 TO N2
110   INPUT X(J)
120   NEXT J
130   A = Y(N2)
140   Y2 = Y(1)
150   FOR L = 1 TO N2
160   IF Y(L) > = A THEN A = Y(L) + 1
170   NEXT L
180   V = 1 -Y2/A
190   N3 = N2-1
200   C2 = 0
210   S2 = 0
220   FOR I = 2 TO N3
230   C2 = C2 + 1
240   C = (-1/X(I))*(LOG((1/V)*(1-(Y(I)/A))))
250   S2 = S2 + C;NEXT I
260   C = S2/C2
270   S = 0
280   S3 = 0
290   Q2 = 0
300   S4 = 0
310   FOR B = 1 TO N2
320   Q2 = Q2 + Y(B)+2
330   S4 = S4 + Y(B)
340   T2(B) = A*(1-V*EXP(-1*C*X(B)))
350   D2(B) = T2(B)-Y(B)
360   S = S + D2(B)
370   S3 = S3 + (D2(B)+2)
380   NEXT B
390   C3 = Q2-(S4+2)/N2
400   R2 = 1-(S3/C3)
410   PRINT"SOLUTION :"
420   PRINT
430   PRINT"YO =";Y2
440   PRINT"V =";V
450   PRINT"C =";C
460   PRINT"A =";A
470   PRINT"SUM DEV =";S
480   PRINT"SSQ =";S3
490   PRINT"R2 =";R2
500   PRINT"DEVIATIONS :"
510   FOR D = 1 TO N2
520   PRINT D2(D)
530   NEXT D
540   P2 = (1-V)*100
550   PRINT"DO YOU WANT NEW VALUES OF YO AND A"
560   INPUTS S$
570   IF S$ = Y$ THEN PRINT"ENTER THEM":INPUT Y2,A;GOTO 190
580   PRINT"DO YOU WANT TO SEE THE PREDICTED VALUES"
590   INPUT M$
600   IF M$ = Y$ THEN GOTO 650
610   PRINT"DO YOU WANT ANOTHER PROBLEM"
620   INPUT B$
630   IF B$ = Y$ THEN GOTC 30
640   END
650   PRINT"C =";C
660   PRINT"V =";V
670   PRINT"A =";A
680   PRINT"PCT SFCY =";P2
690   PRINT"Y VALUES :"
700   FOR G = 1 TO N2
710   PRINT Y(G)
720   NEXT G
730   PRINT"X VALUES :"
740   FOR A = 1 TO N2
750   PRINT X(A)
760   NEXT A
770   GOTO 610
```

Fig. 2. A BASIC program for fitting a Mitscherlich Equation to a set of data points.

```
HOW MANY DATA POINTS DO YOU HAVE?
?
.6
ENTER THE  6 VALUES OF THE DEPENDENT VARIABLE:
?
.45 62 75 83 86 87
ENTER THE  6 VALUES OF THE INDEPENDENT VARIABLE:
?
.0 10 20 30 40 50
SOLUTION: YO =   45.0000 V = 0.4886 C =    0.0646 A =    88.00
SUM DEV =     1.5433 SSQ =          16.9033 R2 = 0.988
DEVIATIONS:  -0.00  3.47  1.20 -1.18 -1.24 -0.70
ENTER NEW VALUES OF YO AND A (OR PRESS RETURN TO STOP):
?
.45 90
SOLUTION: YO =   45.0000 V = 0.5000 C =    0.0562 A =    90.00
SUM DEV =     0.9556 SSQ =           8.1023 R2 = 0.994
DEVIATIONS:   0.0   2.35  0.38 -1.33 -0.75  0.29
ENTER NEW VALUES OF YO AND A (OR PRESS RETURN TO STOP):
?
.45 91
SOLUTION: YO =   45.0000 V = 0.5055 C =    0.0532 A =    91.00
SUM DEV =     1.0616 SSQ =           6.5154 R2 = 0.995
DEVIATIONS:  -0.00  1.97  0.12 -1.33 -0.48  0.78
ENTER NEW VALUES OF YO AND A (OR PRESS RETURN TO STOP):
?
.44 91
SOLUTION: YO =   44.0000 V = 0.5165 C =    0.0543 A =    91.00
SUM DEV =     0.1435 SSQ =           6.2834 R2 = 0.995
DEVIATIONS:  -1.00  1.69  0.13 -1.22 -0.36  0.89
ENTER NEW VALUES OF YO AND A (OR PRESS RETURN TO STOP):
?
.44.5 91
SOLUTION: YO =   44.5000 V = 0.5110 C =    0.0537 A =    91.00
SUM DEV =     0.6041 SSQ =           6.1177 R2 = 0.996
DEVIATIONS:  -0.50  1.83  0.13 -1.27 -0.42  0.83
ENTER NEW VALUES OF YO AND A (OR PRESS RETURN TO STOP):
?
.
DO YOU WANT TO SEE THE PREDICTED VALUES?
.yes
C =   0.0537 V =   0.5110 A =    91.00 PCT SFCY = 48.9
Y VALUES:  44.50  63.83  75.13  81.73  85.58  87.83
X VALUES:   0.0   10.00  20.00  30.00  40.00  50.00
DO YOU WANT TO DO ANOTHER PROBLEM?
.no
R;
```

Fig. 3. An example terminal session using the FORTRAN program to fit a Mitscherlich Equation to a set of data points.

The program begins with a request for the number of levels and the highest level of each independent variable and the number of data points available. A lowest level of zero is assumed. Then the values of x_1, x_2, and y are entered for each data point. The value of A is set by the program at the greatest value of y among those entered. The values for y_{o_1} (the yield for the minimum value of x_1 and the maximum value of x_2) and y_{o_2} (the yield for the minimum value of x_2 and the maximum value of x_1) are assigned the observed values at these respective points. The respective c-values are calculated at the midpoints of each limiting curve; that is, c_1 is obtained from the curve where x_2 is maximum and c_2 is obtained from the curve where x_1 is maximum. The equations used for the calculation of the v-values and the c-values were essentially the same as those used for the one-dimensional case.

It becomes evident at this point that four values of the dependent variable must be present in the data set entered. These include data at the following coordinates: 1) the maximum value of x_1 and the minimum value of x_2, 2) the minimum value of x_1 and the maximum value of x_2, 3) a midpoint value of x_1 and the maximum value of x_2, and 4) a midpoint value of x_2 and the maximum value of x_1. There are no restrictions on the other data points to be included.

The information collected allows the computation of predicted y-values from the equation generated. The student receives the values for A, v_1, v_2, y_{o_1}, y_{o_2}, c_1, c_2, the sum and sum of squares of deviations of the calculated from the observed data, the R^2 value, and the two percentage sufficiency values. The student then has the option of entering new values for any of y_{o_1}, y_{o_2}, c_1, c_2, and A based on the results of the first run. The process can be repeated as many times as desired. Upon termination of the program, an array of the y-values will be typed.

Somewhat more skill is needed to foresee the changes required to improve the fit of the equation in the two-dimensional case. However, most students develop this ability with practice.

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